Mathematical Enrichment

6/2/16

(Number Theory
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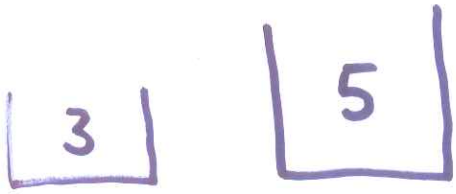

pour exactly 4

$$2\cdot\underline{5} - 2\cdot\underline{3} = 4$$

$$3\cdot\underline{3} - 1\cdot\underline{5} = 4$$

ie. solve

$$x\cdot 3 \pm y\cdot 5 = 4$$

x, y are integers

pour exactly 1 gal:

$$2\cdot\underline{3} - 1\cdot\underline{5} = 1$$

pour exactly 1

Mathematical problem

Solve $5x + 17y = 1$ x, y integers

Eg. $\underline{5}\cdot 7 - \underline{17}\cdot 2 = 1$ $\left(\begin{matrix} x = 7 \\ y = -2 \end{matrix}\right)$

Another $\underline{5}\cdot -10 + \underline{17}\cdot 3 = 1$

$-17\cdot 5$ $+17\cdot 5$

$$7\cdot\underline{5} - 2\cdot\underline{17} = 1$$
$$-17\cdot\underline{5} + 5\cdot\underline{17}$$

$$-10\cdot\underline{5} + 3\cdot\underline{17} = 1$$
$$-17\cdot\underline{5} + \underline{5}\cdot17$$

$$-27\cdot\underline{5} + 8\cdot\underline{17} = 1$$

$$7\cdot\underline{5} - 2\cdot17 = 1$$
$$+17\cdot\underline{5} - 5\cdot17$$

$$24\cdot\underline{5} - 7\cdot17 = 1.$$

$\boxed{6}$ $\boxed{16}$ pow 1 ?

15. Solve $6x + 16y = 1$ x, y <u>integers</u>

No solution since L.H.S. must always be <u>even</u>. (and the R.H.S is <u>not</u>, of course)

$\boxed{6}$ $\boxed{15}$ Pour 1.

$$6x + 15y = 1 \ ? \qquad x, y \text{ integers}$$

<u>No:</u> The left hand side is always a multiple of 3 (divisible by 3)

Since 3 divides 6 and 3 divides 15;

3 is a common divisor of 6, 15.

In fact 3 is the greatest common divisor of 6, 15.

$\cancel{\left(\frac{6}{3} \ \frac{3}{6}\right)}$

$$3 \mid 6 \quad \text{and} \quad 3 \mid 15$$

$$3 = \text{gcd}(6, 15)$$

or simply

$$3 = (6, 15)$$

<u>To summarize:</u> $\boxed{m}$ $\boxed{n}$ pour ℓ

<u>Mathematically</u>: Solve

$$xm + yn = \ell$$ with x, y integers

We've just observed.

If d is a common divisor of m and n, then <u>no solution</u> if $d \nmid l$

In particular, no solution if $g \nmid l$ where $(m,n) = g$.

<u>Question</u> Let $g = (m,n)$.

Is g obtainable? If so, how?

<u>Answer</u>: Yes! by <u>Euclid's algorithm</u>.

We use the following <u>basic principle</u>

Let a, b be any two integers.

Suppose $b = t \cdot a + r$ where t, r are <u>integers</u>

Then $(a,b) = (a,r)$

<u>Why?</u> Any common divisor of a, r is also a common divisor of a, b:

$d \mid a$ and $d \mid r \Rightarrow d \mid t \cdot a + r = b \Rightarrow d \mid a, b.$

Similarly, since $r = t \cdot a - b$

any common divisor of a, b is also a common divisor of a, r

How does this help?

(Toy) Example:

$a, b = \underset{a}{21}, \underset{b}{51}$

$$\boxed{\begin{aligned} \underset{b}{51} &= 2 \cdot \underset{a}{21} + \underset{r}{9} \quad (1) \\ 21 &= 2 \cdot 9 + 3 \quad (2) \end{aligned}}$$

$(\; 9 = 3 \cdot 3 + \boxed{0} \;) \; - \; 3 \mid 9$

$\Rightarrow$

$(9,3) = 3$

$\|$

$(21, 9)$

$\|$

$(51, 21)$

We want to solve $\boxed{21x + 51y = 3}$

(2) $\quad 3 = 21 - 2 \cdot 9$

$\quad\;\; = 21 - 2 \cdot (51 - 2 \cdot 21)$

$\Rightarrow \boxed{3 = 5 \cdot 21 - 2 \cdot 51}$ $\quad x = 5$, $y = -2$

$$5\cdot 21 - 2\cdot 51 = 3$$

$$(-12\cdot 21 + 5\cdot 51 = 3)$$

$$-51\cdot 21 + 21\cdot 51$$

$$-46\cdot 21 + 19\cdot 51 = 3.$$

$\boxed{1007}$ $\boxed{703}$

What's the gcd?
How do we obtain it?

$$1007 = 1\cdot 703 + 304$$
$$703 = 2\cdot 304 + 95$$
$$304 = 3\cdot 95 + 19$$

$$95 = 5\cdot 19$$

So $19 = (95,19) = (304,95) = (703,304) = (1007,703)$

<u>Pour 19 gallons:</u>

$$\begin{aligned} 19 &= 304 - 3\cdot 95 \\ &= 304 - 3\cdot(703 - 2\cdot 304) \\ &= 7\cdot 304 - 3\cdot 703 \\ &= 7\cdot(1007 - 703) - 3\cdot 703 = 7\cdot 1007 - 10\cdot 703 \end{aligned}$$

How about $38 = 2 \cdot 19$?

$$7 \cdot 1007 - 10 \cdot 703 = 19$$

Multiply both sides by 2:

$$14 \cdot 1007 - 20 \cdot 703 = 38$$

Conclusion Given m and n $\left\{\begin{array}{l}\text{gallons}\\ \text{litres...}\end{array}\right.$

l is obtainable if and only if

l is a multiple of $g = (m,n)$.

(It follows we can obtain any (positive) amount l $\iff$ $(m,n) = 1$).

$\boxed{\text{Theorem } (m,n) = 1 \iff \text{we can solve } xm + yn = 1 \text{ in integers } x, y.}$

Some problems

1. $\boxed{17}$ $\boxed{57}$ pour exactly 1 gallon.

2. $\boxed{437}$ $\boxed{986}$

What is the smallest amount that can be obtained? How?

i.e. Find $(437, 986) = g$

and find integers x, y such that

$$437x + 986y = g.$$

3. (I.M.O Romania 1959)

Prove that the fraction $\dfrac{21n+4}{14n+3}$

is irreducible for any natural number n.

(natural number : positive integer

irreducible $\to$ top and bottom have no common divisor except 1)

4. $\boxed{3c}$ $\boxed{5c}$ stamps. Lots of them.

What amounts can be obtained using only 3c and 5c stamps?..